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Lower bound for LCF lifetime and its application to safe design of elastic viscoplastic structures

B. Druyanov *, I. Roman

Graduate School of Applied Science, Hebrew University of Jerusalem, Givat Ram, P.O. Box 6845, 91904 Jerusalem, Israel

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Abstract

The theory of shakedown is applied to obtain an upper estimation of LCF lifetime of structures. A model of elastic viscoplastic material similar to the Perzyna one with isotropic strain hardening and isotropic damage is adopted. Assumptions: viscoplastic strain rate is proportional to the access of the yield function over zero; the rate of damage evolution is equal to a function of hardening and damage parameters with the coefficient of fluidity, as a factor of proportionality; damage process is coupled with the viscoplastic deformation process; the hardening parameter is equal to accumulated viscoplastic deformation. The yield surfaces form a family of self-similar surfaces with the diameter as the parameter. The shakedown condition of the Melan type is formulated relatively to the initial yield surface. Features of the stress path lead to an equation with min–max problem of the mathematical programming in the left side, which determines a safe value of the virtual residual stress. The equation provides an opportunity to compute the maximal value of the strain hardening parameter possible under the prescribed loading program. This value allows to obtain an upper estimate to safe work time of the structure, which results in a sufficient condition of the structure integrity during the prescribed time period. An example of the developed theory application to resolve various problems arising from designing of structures is considered.

Keywords: Shakedown; Hardening; Damage; Viscoplasticity; Low cycle fatigue life; Safe design

* Corresponding author. Tel.: +972 2 642 69 37; fax: +972 2 566 38 78.
E-mail address: borisu@bezeqint.net (B. Druyanov).

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1. Introduction

The problem of direct estimation of low cycle fatigue lifetime is still topical. A large number of papers devoted to this problem confirm this assertion. However, as far as the authors are informed, pure theoretical methods based on Solid Mechanics are absent.

The paper aims at developing a theoretical method for directly estimating the LCF lifetime of metallic structures subjected to static cyclic mechanical loading under elevated temperature. Under these conditions, the strength of the structural materials in the domain of irreversible deformation depends on the rate of deforming. Such material behavior is usually modeled by viscoplastic constitutive material models. (See, for example, Chaboche, 1977; Lemaitre and Chaboche, 1990).

Polizzotto (1995) considered structures of elastic viscoplastic material models with dual internal variables, thermodynamic potential and temperature-dependent material parameters, as subjected to variable mechanical and thermal loading. The conditions of existence of steady-state responses (elastic/inelastic shakedown) of such structures and some of their features were investigated.

In this paper, a variant of elastic viscoplastic material model by Perzyna (1974) generalized to account for isotropic strain hardening and damage is taken as the base for investigation. As in Perzyna, the rate of viscoplastic strain is assumed to be proportional to the excess of the current yield function value over zero. Due to this property, the notion of elastic shakedown is readily extended to the adopted material model, as the viscoplastic deformation does not occur, if the current yield function is negative or equal to zero.

The theory of shakedown is applied to resolve the problem to be sought. An extension of the static elastic shakedown theorem to damaged materials was developed in Druyanov and Roman (2002). Here, a novel formulation of the theorem, as applied to adopted material model, is given.

Damage process is assumed to be coupled with viscoplastic deformation process: it is in progress, if and only if the viscoplastic deformation is progressing. The local failure of the material occurs, when the damage parameter reaches its critical value, which is considered as a material parameter. The corresponding time instant could be named as the critical time.

The rate of damage evolution is taken as equal to a function of hardening and damage parameters with a material parameter of the dimension 1/s (the coefficient of fluidity), as a factor of proportionality. Integration of the damage evolutionary equation over time from an initial time to the critical time results in a lower bound for the critical time. The bound depends on the maximal value of the hardening parameter possible under the given loading program. This value corresponds to the maximal admissible value of the damage parameter, which is assumed equal to its critical value.

The method proposed in Druyanov and Roman (1997) provides an opportunity for direct determination of a connection between maximal values of the hardening and damage parameters possible under the given loading program. The maximal admissible value of the damage parameter is assumed equal to its critical value. This assumption determines the maximal value of the hardening parameter, which, in turn, allows determining a lower estimation of the critical time.

A structure saves its integrity, if it is prescribed work time (i.e., the time period during which the structure is to retain its integrity) is less then this estimation. This is a sufficient condition for integrity during the prescribed work time, which was sought.

The problem of safe designing of structures subjected to cyclic loading is particularly in determining such values of the structure parameters, which would guarantee integrity of the structure during a prescribed work time period. This implies that the LCF lifetime of the structure elements should be not less than the prescribed work time period. The findings obtained in the paper provide an opportunity of directly resolving this problem like it was done for elastic–plastic damaged structures in Druyanov and Roman (in press).

An example of application of the developed methods is given.

The obtained findings can be extended to the case of materials with temperature-dependent properties and thermo-mechanical loading by the method proposed in Druyanov and Roman (2005).

2. Constitutive material model

A model of elastic viscoplastic material like Perzyna (1974) with isotropic strain hardening and isotropic viscoplastic damage is taken. The consideration is restricted by purely mechanical theory. Inertia forces and temperature effects are neglected. The deformation is assumed to be small, so that the total strain tensor can be decomposed into elastic ($\boldsymbol{\varepsilon}^{e}$) and viscoplastic ($\boldsymbol{\varepsilon}^{p}$) parts.

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{e} + \boldsymbol{\varepsilon}^{p}. \tag{1}$$

2.1. Thermodynamics

Let $\mathbf{L}$ denote the current (damaged) value of the elastic compliance tensor with its ordinary symmetry, and $\mathbf{C} = \mathbf{L}^{-1}$ denote the corresponding stiffness tensor. Either of these tensors can be taken for damage variable.

Denote through χ the isotropic strain hardening parameter, which is taken equal to the accumulated viscoplastic deformation: $\dot{\chi} = (\dot{\boldsymbol{\varepsilon}}^{p} : \dot{\boldsymbol{\varepsilon}}^{p})^{1/2}$. The local damaged Helmholtz free energy function is formulated as

$$\Psi(\boldsymbol{\varepsilon}^{e}, \chi, \mathbf{C}) \equiv \tfrac{1}{2}\boldsymbol{\varepsilon}^{e} : \mathbf{C} : \boldsymbol{\varepsilon}^{e} + \Psi_{p}(\chi), \tag{2}$$

where the first term in the right side represents the elastic part of the free energy, and the term $\Psi_{p}(\chi)$ is the free energy stored at the micro level due to strain hardening. As in Lemaitre (1992), the effect of damaging on this part of the free energy is neglected.

The Clausius–Duhem inequality $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} \geqslant 0$, where $\boldsymbol{\sigma}$ denotes the nominal stress tensor, should be valid for any thermo-mechanical process. Employing the Coleman and Gurtin (1967) arguments, we arrive at the elastic strain–stress relation

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}^{e} \tag{3}$$

and the dissipative inequalities

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{p} - \frac{\partial \Psi_{p}}{\partial \chi}\dot{\chi} \geqslant 0, \tag{4}$$

$$\boldsymbol{\varepsilon}^{e} : \dot{\mathbf{C}} : \boldsymbol{\varepsilon}^{e} \leqslant 0. \tag{5}$$

In order for inequality (5) to be valid for any deformation process, the quadratic function $\boldsymbol{\varepsilon}^{e} : \dot{\mathbf{C}} : \boldsymbol{\varepsilon}^{e}$ should not be positive. This condition is fulfilled, if all eigenvalues of the matrix $\dot{\mathbf{C}}$ are not positive.

As $\mathbf{L} : \mathbf{C} = \mathbf{I}$, then $\boldsymbol{\varepsilon}^{e} : \dot{\mathbf{C}} : \boldsymbol{\varepsilon}^{e} = -\boldsymbol{\sigma} : \dot{\mathbf{L}} : \boldsymbol{\sigma}$. Consequently,

$$\boldsymbol{\sigma} : \dot{\mathbf{L}} : \boldsymbol{\sigma} \geqslant 0 \tag{6}$$

i.e., the eigenvalues of the quadratic function $\boldsymbol{\sigma} : \dot{\mathbf{L}} : \boldsymbol{\sigma}$ should not be negative (Druyanov and Roman, 2003).

Inequalities (5) and (6) are transformed into equalities only in the absence of damaging.

In the case of isotropic damage, the current (damaged) value of the elastic stiffness tensor $\mathbf{C}$ is defined as $\mathbf{C} = (1 - \Delta)\mathbf{C}_{un}$, where $\mathbf{C}_{un}$ denotes the undamaged value of $\mathbf{C}$, and Δ is the isotropic damage parameter. Analogously $\mathbf{L} = \mathbf{L}_{un}/(1 - \Delta)$. In this case, the above inequalities lead to the known conclusion that the rate of damage parameter ($\dot{\Delta}$) is non-negative (Lemaitre, 1992).

Denote through $\varDelta_c$ the critical value of the damage parameter. If $\varDelta \to \varDelta_c$, the structure material fails at the point under consideration.

2.2. Viscoplasticity

Let $\bar{\xi} = \bar{\Phi}(\bar{\boldsymbol{\sigma}}, \chi)$ denote the yield function where $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1-\varDelta)$ is the effective stress. It is assumed $\bar{\xi} = \bar{\Phi}(0, \chi) < 0$. The function $\bar{\Phi}(\bar{\boldsymbol{\sigma}}, \chi)$ is presupposed regular, convex, increasing in components of $\bar{\boldsymbol{\sigma}}$, decreasing in χ up to the hardening saturation point (χ_h), which is a material parameter. It corresponds to the ultimate tension point at the strain–stress diagram. Starting from this point, material softening begins.

Yield surface $\left(\xi = \Phi\left(\frac{\boldsymbol{\sigma}}{1-\varDelta}, \chi\right) = 0\right)$ depends on two parameters: χ and $\varDelta$. A change in any of them causes a homothetic transformation of the yield surface. A specification of values of χ and $\varDelta$ define the yield surface and its diameter. Oppositely: a specification of the diameter defines a dependence between χ and $\varDelta$. So, the yield surfaces form one-parametrical family of surfaces with the diameter of yield surface ($D(\mathbf{x}, t)$) as the parameter. The diameter is a known function of $\varDelta$ and χ, decreasing in $\varDelta$ and increasing in χ. Consequently, the equation of yield surface in the nominal stress space can be recast as $\xi = \Phi(\boldsymbol{\sigma}, D) = 0$. The diameter could be considered as the current doubled effective yield stress. The current values of the diameter are determined by deformation process.

Take the yield condition of the Mises type, for example. Its equation in nominal stress may be written in the form: $\Phi = f(\boldsymbol{\sigma}) - (1-\varDelta)\kappa(\chi) = 0$ where $f(\boldsymbol{\sigma})$ is a uniform function of the first rank in $\boldsymbol{\sigma}$, and $\kappa(\chi)$ is an yield stress. The quantity $(1-\varDelta)\kappa(\chi)$ is proportional to the diameter of the Mises cylinder. So the yield surface equation can be rewritten in the form: $\Phi = f(\boldsymbol{\sigma}) - aD = 0$ where a is a number.

The rate of viscoplastic strain is assumed to be proportional to the access of the yield function value over zero:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = v\langle\bar{\xi}\rangle \frac{\partial \bar{\xi}}{\partial \bar{\boldsymbol{\sigma}}} = \frac{1}{2} v \frac{\partial \langle\bar{\xi}\rangle^2}{\partial \bar{\boldsymbol{\sigma}}}, \tag{7}$$

where $\langle\bar{\xi}\rangle = \frac{1}{2}(\bar{\xi} + |\bar{\xi}|) = \begin{bmatrix} \bar{\xi}, & \text{for } \bar{\xi} > 0 \\ 0, & \text{for } \bar{\xi} 0 \end{bmatrix}$ and v is the coefficient of fluidity—a material parameter of the dimension 1/s.

Any level yield surface $\bar{\xi} = \bar{\Phi}_\alpha(\alpha\bar{\boldsymbol{\sigma}}, \chi) = 0$, where α is a positive number, can be obtained from the yield surface $\bar{\xi} = \bar{\Phi}(\bar{\boldsymbol{\sigma}}, \chi) = 0$ by a similarity transformation. Because, the yield surface is assumed to be convex, all level surfaces are also convex.

Rate of the viscoplastic strain ($\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$) at a point of the space $\bar{\boldsymbol{\sigma}}$ is normal to the level surface $\bar{\xi} = \bar{\Phi}(h\bar{\boldsymbol{\sigma}}, \chi) = 0$ passing through this point.

Let $\hat{\bar{\boldsymbol{\sigma}}} = \hat{\boldsymbol{\sigma}}/(1-\varDelta)$ denote a safe effective stress: $\bar{\xi} = \bar{\Phi}(\hat{\bar{\boldsymbol{\sigma}}}, \chi) \leqslant 0$, i.e., the corresponding stress point is on the yield surface or in the interior of it, and $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1-\varDelta)$ be an active stress such that $\bar{\xi} = \bar{\Phi}(\bar{\boldsymbol{\sigma}}, \chi) > 0$, i.e., the stress point corresponding to $\bar{\boldsymbol{\sigma}}$ is in the exterior of the yield surface. The value of the damage parameter $\varDelta$ is assumed actual. Then, in virtue of the convexity of the level surfaces, the inequality holds $(\bar{\boldsymbol{\sigma}} - \hat{\bar{\boldsymbol{\sigma}}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \geqslant 0$. The equality takes place if and only if $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = 0$.

Transferring to the nominal stress tensor provides

$$(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \geqslant 0. \tag{8}$$

The unloading and subsequent reloading processes are assumed purely elastic. Therefore, during these processes, the elastic stiffness tensor $\mathbf{C}$ and the elastic compliance tensor $\mathbf{L}$ save their current (damaged) values, which they had at the start of unloading. Consequently, the stress tensor can be decomposed as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho}, \tag{9}$$

where $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x}, t)$ represents the current purely elastic response of the structure under consideration to the current boundary conditions, and $\boldsymbol{\rho}(\mathbf{x}, t)$ is the current tensor of residual stress. A way of directly computing the function $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x}, t)$ accounting for the change in elastic moduli due to damaging is considered in Section 6. The ordinary decomposition of the actual strain tensor is valid

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{E}} + \boldsymbol{\varepsilon}^{\mathrm{re}} + \boldsymbol{\varepsilon}^{\mathrm{p}}, \tag{10}$$

where $\boldsymbol{\varepsilon}^{\mathrm{E}}$ is the elastic strain corresponding to $\boldsymbol{\sigma}^{\mathrm{E}}$:$\boldsymbol{\varepsilon}^{\mathrm{E}} = L$:$\boldsymbol{\sigma}^{\mathrm{E}}$, $\boldsymbol{\varepsilon}^{\mathrm{re}}$ is the elastic part of the residual strain tensor: $\boldsymbol{\varepsilon}^{\mathrm{re}} = \mathbf{L}$:$\boldsymbol{\rho}$, and $\boldsymbol{\varepsilon}^{\mathrm{p}}$ is its viscoplastic part.

2.3. Shakedown

The rate of viscoplastic strain is proportional to the excess of the yield function over zero. Due to this property, the notion of shakedown is readily extended to the adopted material model. Suitable shakedown conditions are given in Section 7.

2.4. Damage

The damage process is assumed coupled with the process of viscoplastic deformation, i.e., the damage process starts and ceases simultaneously with the start and cessation of the viscoplastic deformation process.

It is assumed that no residual strain and stress are induced by damage.

The rate of damage growth is defined by the equation, which is similar to the one proposed by Ju (1989).

$$\dot{\Delta} = v g(\chi, \Delta) H(\dot{\chi}), \tag{11}$$

where $H(\dot{\chi})$ is the step function: $H(\dot{\chi}) = 1,\ \text{if}\,\dot{\chi} > 0, H(\dot{\chi}) = 0,\ \text{if}\,\dot{\chi} \leqslant 0$; and $g(\chi, \Delta)$ is a given material function increasing with its arguments.

According to (11),

$$\Delta = v \int_{t_0}^{t} g(\chi, \Delta)\, \mathrm{d}t. \tag{12}$$

3. Sufficient condition of survivability

If shakedown condition is fulfilled, then the deformation process reaches eventually a stationary stage, at which damaging ceases. However, the structure can fail at the transient stage of deformation process, if the damage parameter reaches its critical value (Δ_{c}). Nevertheless, it is possible to believe formally that the deformation process continues after this point.

Let the damage parameter reaches its critical value (Δ_{c}) at the transient stage. In the opposite case, the structure will not fail. Let t_{c} denotes the corresponding time instant named as the critical time. Corresponding value of χ is denoted by χ_{m}. This is the maximal value of the hardening parameter, which it can acquire until the structure fails, i.e., for the period from t_0 to t_{c}, where t_0 denotes the initial time instant of deformation process. The duration of time period $t_0 \leqslant t \leqslant t_{\mathrm{c}}$ is the low cycle fatigue lifetime. Since at this period $\chi \leqslant \chi_{\mathrm{m}}$, then

$$\Delta_{\mathrm{c}} \leqslant v \int_{t_0}^{t_c} g(\chi_{\mathrm{m}}, \Delta_{\mathrm{c}})\, \mathrm{d}t = v g(\chi_{\mathrm{m}}, \Delta_{\mathrm{c}})(t_{\mathrm{c}} - t_0). \tag{13}$$

Consequently,

$$\frac{\varDelta_c}{vg(\chi_m, \varDelta_c)} \leqslant t_c - t_0. \tag{14}$$

The structure saves its integrity during the designed work time period ($t_w - t_0$), if t_w is less than the critical time t_c: $t_w < t_c$. This condition is fulfilled, if

$$t_w - t_0 < \frac{\varDelta_c}{vg(\chi_m, \varDelta_c)}. \tag{15}$$

This is a sufficient condition of local integrity during the work time period $t_w - t_0$ in the case, where the damage parameter reaches its critical value at the transient stage.

Condition (15) could be named as the condition of structure survivability.

A way of directly computing χ_m is considered in Section 5.

The overall integrity of the structure for the period $t_w - t_0$ will be observed, if inequality (15) is fulfilled for the minimal value of the fraction in its right side over all the structures.

4. Features of post-adaptation stage of deformation process

If the condition of shakedown (Section 7) is satisfied, and if the opportunity of failure is ignored, then, formally speaking, the deformation process reaches eventually the stationary (post-adaptation) stage where no irreversible deformation and damage occur. Time-independent values of the residual stress tensor $\boldsymbol{\rho}_s$, the damage parameter $\varDelta_s$, the hardening parameter χ_s, and the diameter D_s are the characteristics for this stage. These values and the corresponding yield surface are named as the limit ones. Because of this circumstance, the nominal stress $\boldsymbol{\sigma}$ is used for further considerations.

At the post-adaptation stage, the actual representative stress point in the stress space $\boldsymbol{\sigma}$ reaches the yield surface repeatedly, but the stress does not induce a progress in damaging and viscoplastic deformation, and the limit yield surface does not change. This is possible, if the stress path does not exit out of the yield surface, and either some parts of the stress path $\boldsymbol{\sigma}(\mathbf{x}, t)$ are placed on the yield surface, or the stress path touches it at some isolated points. In particular, this is valid at the time instants t^* corresponding to the beginning of unloading. These time points will be named as the departure instants.

At these instants, the stress satisfies the equation of the yield surface. Thus, the following equation is valid:

$$\xi = \xi^* = \Phi(\boldsymbol{\sigma}(\mathbf{x}, t^*), D_s(\mathbf{x})) = 0. \tag{16}$$

Because of the cyclic nature of loading, during the post-adaptation stage, the points of local maximum of the yield function are situated either in the interior of the yield surface or on it. According to the assumption, $\xi < 0$ in the first case and $\xi = 0$ in the second case. Hence, the departure points are the points of absolute maximum of the yield function with respect to t.

Employing the decomposition $\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}^E(\mathbf{x}, t) + \boldsymbol{\rho}(\mathbf{x})$, Eq. (16) can be reduced to the form

$$\xi(\mathbf{x}) = \xi^*(\mathbf{x}) = \Phi(\boldsymbol{\sigma}^E(\mathbf{x}, t^*) + \boldsymbol{\rho}_s(\mathbf{x}), D_s(\mathbf{x})) = 0. \tag{17}$$

The function

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}^E(\mathbf{x}, t) + \boldsymbol{\rho}_s(\mathbf{x}) \tag{18}$$

determines the limit stress path at the point $\mathbf{x}$ of the structure in the stress space $\boldsymbol{\sigma}$ at the post-adaptation stage.

5. Estimation of the maximal value of hardening parameter

For a wide class of loading, the features of the post-adaptation stage provide an opportunity to derive directly, i.e., without a detailed investigation of the deformation process, a connection between the possible limit values of damage, hardening parameter, and residual stress. The connection allows obtaining an upper estimate for the values of the hardening parameter possible for the given loading program.

Hereafter, in this section, only the post-adaptation stage of deformation is considered, so that the subscript “s” is omitted.

Due to cyclic nature of loading, stress path (18) has a number of apexes, which are specified by the loading program. As ξ is assumed a non-decreasing function of the stress tensor components, the local and absolute extremums of the yield function correspond to the apexes of the stress path.

The local extremums of the yield function ξ are located in the interior of the yield surface and are negative. The yield function reaches the absolute maximum values which are equal to zero at the departure points, i.e., these maximum values of ξ are equal to each other. This situation can be modeled, if one considers that the absolute maximums of the yield function to be minimal. As a result, the following specification of Eq. (17) is arrived (Druyanov and Roman, 1997)

$$\xi_{\mathrm{m}}(\mathbf{x}) \equiv \min_{\boldsymbol{\rho}} \max_{t} \xi(\mathbf{x}) \equiv \min_{\boldsymbol{\rho}} \max_{t} \Phi(\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x}, t) + \boldsymbol{\rho}(\mathbf{x}), D(\mathbf{x})) = 0. \tag{19}$$

This min–max problem should be resolved for fixed values of $\mathbf{x}$ and D.

Residual stress tensor $\boldsymbol{\rho}$ has to satisfy the equilibrium equations and zero boundary conditions: $\boldsymbol{\rho} \cdot \mathbf{n} = 0$ at the part of the solid surface S_{p}, where tractions are prescribed, $\mathbf{n}$ is the unit vector of the external normal to S_{p} and $\mathbf{a} \cdot \mathbf{b} = a_i b_i$.

A direct way of computing $\boldsymbol{\sigma}^{\mathrm{E}}$ is considered in Section 6.

Remark. The equilibrium equations can be satisfied by introducing the stress functions (Timoshenko and Goodier, 1951), which are defined by the min–max problem in the left of (19). For example, in the case of plane strain and the Mises yield condition the min–max problem is reduced to a boundary-value problem for a non-uniform hyperbolic equation in partial derivatives of the second rank (Druyanov and Roman, 2002).

A solution of the min-max problem in the left side of (19) provides us with the values $\xi_{\mathrm{m}}(\mathbf{x})$, $\boldsymbol{\rho}_{\mathrm{m}}(\mathbf{x})$, and $t^*(\mathbf{x})$ as functions of D. In turn, Eq. (19) determines the diameter of the yield surface $\xi_{\mathrm{m}} = 0$, which will be denoted D_{m}. Besides $\boldsymbol{\rho}_{\mathrm{m}}$ determines a certain position of the stress path $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x}, t) + \boldsymbol{\rho}_{\mathrm{m}}(\mathbf{x})$ with regard to the yield surface.

Take the yield condition $\Phi = f(\boldsymbol{\sigma}) - D = 0$, for example. The value of $f(\boldsymbol{\sigma})$ is determined by the solution of Eq. (19): In turn, it determines the diameter of the yield surface.

The quantity $\boldsymbol{\rho}_{\mathrm{m}}$ gives a minimal value to the function $\max \xi$. As $\min \max \xi$ is equal to zero, any change in $\boldsymbol{\rho}_{\mathrm{m}}$ provides a positive value to $\max \xi$, i.e., it shifts the stress path in such a way that at least one of its apexes falls outside the limit yield surface. Hence, if at least two apexes of the stress path are placed at the yield surface, then they coincide with the ends of the chord of the maximal length, i.e., with the ends of yield surface diameter. As there are no apexes outside the yield surface, these two apexes are also the ends of the diameter of the stress path. Hence, the solution of Eq. (19) defines the limit yield surface in such a way that its diameter coincides with the diameter of the stress path. Thus, the solution of Eq. (19) provides us with the minimal value of the diameter of the yield surface possible for the prescribed loading program.

Eq. (19) defines D and consequently, Δ as a function of χ: $\Delta = \varphi(\chi)$. Because $\bar{\Phi}(\bar{\boldsymbol{\sigma}}, \chi)$ is an increasing function of $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1 - \Delta)$ and a decreasing function of χ at the interval $\chi_0 \leqslant \chi < \chi_{\mathrm{h}}$, Eq. (19) defines Δ at this interval as an increasing function of χ. At the interval $\chi > \chi_{\mathrm{h}}$ this function is a decreasing one. Thus, the value of Δ corresponding to the hardening saturation point (χ_{h}) is the maximal value of the damage parameter possible under the prescribed loading program. This value is denoted as Δ_h. Obviously, its value depends on the loading program.

If $\Delta_h < \Delta_c$, then the deformation process under consideration reaches the stationary stage successfully, i.e., without failure at the point under consideration. This is a sufficient condition of shakedown. In the opposite case ($\Delta_h > \Delta_c$), local failure is possible.

Suppose $\Delta_h > \Delta_c$. If the shakedown condition is satisfied, then, formally speaking, the deformation process reaches the stationary stage irrespective of whether the damage parameter has acquired the value of Δ_c or not. During deformation, the damage parameter increases. Its maximal safe value is Δ_c. Thus, the maximal safe value of χ possible under the given loading program is determined by Eq. (19) at $\Delta = \Delta_c$. Denote this value as χ_m. There are two values of χ corresponding to Δ_c in the case $\Delta_h > \Delta_c$, which correspond to ascending and descending branches of the curve $\Delta = \varphi(\chi)$. The relevant value should be taken from the ascending part of this curve, because Δ is an increasing parameter. In this case $\chi_m < \chi_h$.

6. Direct way to compute the purely elastic response

Eq. (19) depends on the function $\boldsymbol{\sigma}^E(\mathbf{x}, t^*)$, which represents the actual (damaged) purely elastic response of the structure at the departure instants to the prescribed loading program. This function is determined by resolving the elastic boundary-value problem for the structure under consideration at the time instants t^* for the corresponding boundary conditions, and Hook's law $\boldsymbol{\varepsilon}^E = \mathbf{L}{:}\boldsymbol{\sigma}^E$ where $\mathbf{L}$ is the current (damaged) value of the elastic compliance tensor: $\mathbf{L} = \mathbf{L}_0/(1 - \Delta)$. The values of Δ and χ at $t = t^*$ could be calculated by means of detailed investigation of the entire deformation process. To avoid this way, a direct method to calculate $\boldsymbol{\sigma}^E(\mathbf{x}, t^*)$ is considered below (Druyanov and Roman, 2002).

At the departure instants $t = t^*$ the function $\boldsymbol{\sigma}^E(\mathbf{x}, t^*) + \boldsymbol{\rho}(\mathbf{x})$ satisfies the equation of yield surface. This feature gives us a chance to obviate a detailed investigation of the deformation process, and to compute $\boldsymbol{\sigma}^E(\mathbf{x}, t^*)$ by means of resolving a boundary-value problem for the system of the elasticity equations supplemented with the equations $\mathbf{C} = \mathbf{C}_0(1 - \Delta(\mathbf{x}))$ and (19). This system of equations may be named as the basic system. Its solution provides the values of $t^*(\mathbf{x})$, $\boldsymbol{\rho}_m(\mathbf{x})$ and D_m aside from $\boldsymbol{\sigma}^E(\mathbf{x}, t^*)$.

A solution of the min–max problem in the left of (19) determines the value of $\boldsymbol{\rho}$ such that the stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\rho}$ is safe for any value of time, i.e., it is either in the interior of the yield surface, or on it. According to the shakedown condition, this is a sufficient condition of shakedown (see Section 7).

7. Shakedown condition for viscoplastic structures

The Koiter theorem extended to the structures of elastic viscoplastic material is formulated in a classic manner: if there exists a time independent virtual residual stress field $\hat{\boldsymbol{\rho}}(\mathbf{x})$ such that the virtual decomposition $\hat{\boldsymbol{\sigma}}(\mathbf{x}, t) = \boldsymbol{\sigma}^E(\mathbf{x}, t) + \hat{\boldsymbol{\rho}}(\mathbf{x})$ satisfies the yield inequality

$$\Phi\left(\frac{\hat{\boldsymbol{\sigma}}(\mathbf{x}, t)}{1 - \Delta(\mathbf{x}, t)}, \chi(\mathbf{x}, t)\right) \leqslant 0 \tag{20}$$

for $t \geqslant 0$, then the total energy dissipation is bounded.

Hence, the virtual stress path $\hat{\boldsymbol{\sigma}}(\mathbf{x}, t) = \boldsymbol{\sigma}^E(\mathbf{x}, t + \hat{\boldsymbol{\rho}}(\mathbf{x})$ should be either in the interior of the current yield surface, or touch it.

Boundedness of the total dissipated energy can be taken for the necessary and sufficient condition of shakedown (Debordes and Nayroles, 1976).

In the process of deformation, the current yield surface changes its sizes due to hardening and damaging. All current yield surfaces are similar to each other. It is supposed that initially the diameter of the yield surface grows (the stage of hardening), and then it decreases (stage of softening). The maximal value of the diameter depends on the prescribed loading program and could be found by a detailed analysis of the deformation process. The only yield surface, whose diameter is known in advance, is the initial yield surface.

The above theorem demands determinating the virtual residual stress $\hat{\boldsymbol{\rho}}$. Eq. (19) gives a regular way to determine it. Note that Eq. (19) is valid at $t = t^*$. At these time instants, the yield function reaches its maximal value equal to zero. Hence, the values of the yield function for $t \neq t^*$ are negative. As mentioned in the previous section, Eq. (19) determines the yield surface of the minimal diameter (min yield surface), which is possible for the prescribed loading program. Thus, the stress path determined by Eq. (19) is in the interior of the minimal yield surface, otherwise touches it. Consequently, the stress path is also in the interior of subsequent yield surfaces.

This assertion is valid not only at the stage of hardening but also at the part of the softening stage adjacent to the point of maximal hardening until the current yield surfaces embrace the min one.

This conclusion provides an opportunity to formulate a sufficient condition of shakedown as follows. If the min yield surface is in the interior of the initial yield surface, or coincides with it, then the structure under consideration will shake down to the prescribed loading program. This condition can be expressed by the inequality:

$$D_{\min} \leqslant D_0, \tag{21}$$

where $D_{\min}$ denotes the diameter of the min yield surface and D_0 denotes the diameter of the initial yield surface.

Obviously, in the above shakedown condition, the initial yield surface plays the role of a sanctuary (Nayroles and Weichert, 1993).

The right side of inequality (21) is a given quantity. The value of the left side is determined by Eq. (19).

As applied to the Mises yield surface $\Phi = f(\boldsymbol{\sigma}) - (1 - \Delta)\kappa(\chi) = 0$: $D_0 = (1 - \Delta_0)k(\chi_0)$, $D_{\min} = f(\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x}, t^*) + \boldsymbol{\rho}_{\mathrm{m}}(\mathbf{x}))$. So, in this case, the shakedown condition is expressed by the inequality: $f(\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x}, t^*) + \boldsymbol{\rho}_{\mathrm{m}}(\mathbf{x})) \leqslant (1 - \Delta_0)k(\chi_0)$, where $\Delta_0(\mathbf{x})$ and $\chi_0(\mathbf{x})$ are the given initial values of the damage and hardening parameters correspondingly, and $\boldsymbol{\rho}_{\mathrm{m}}(\mathbf{x})$ is the residual stress determined by Eq. (19).

8. Direct safe design

The objective of safe structural design as applied to structures of viscoplastic material is to determine the interval of safe values of a design parameter $\beta(\mathbf{x})$, i.e., such values, for which the structure in design both adapts itself to the given loading program and saves its integrity during the prescribed work time period t_{w}. The method proposed below is an extension of the method developed in Druyanov and Roman (in press) to adopted material model structures.

If β is of geometrical nature, the boundary conditions and, consequently, the function $\boldsymbol{\sigma}^{\mathrm{E}}$ depends on it. If β is a material parameter, then not only $\boldsymbol{\sigma}^{\mathrm{E}}$ but also the yield function depends on it. Also β can be a complex parameter representing a set of parameters.

The possible values of β have to satisfy a priori requirements arising from the nature and service conditions of the structure. These requirements can be expressed by a set of inequalities. For example, a geometrical parameter cannot be negative. The interval of admissible values of β can be found by comparing the safe interval for β with the above-mentioned a priori inequalities.

It is assumed that the loading program is prescribed. In the case where the loading program is unknown and only the bounds for variation of applied loads are given, the proposed method provides a necessary condition for shakedown and integrity during the prescribed work period t_{w}.

The condition of shakedown is expressed by inequality (19)/(20), which depends on β. The quantities χ_{m} and $\frac{\Delta_{\mathrm{c}}}{v\chi_{\mathrm{m}}}$ also depend on β. Therefore it is possible, in principle, to satisfy condition of integrity (15) and shakedown condition (19) by a proper choice of β. The satisfaction of these inequalities guarantees the integrity of the structure during the prescribed time period.

The algorithm of designing can be sketched as follows. In the first place, the condition of integrity (15) has to be specified. To that end, the boundary-value problem for the basic system of equations (the system of elasticity equations supplemented with equations $\mathbf{C} = \mathbf{C}_0(1 - \Delta)$ and (19)) has to be resolved under $\Delta = \Delta_c$ and the given boundary conditions. The solution of this system provides the maximal value of the strain hardening parameter: $\chi = \chi_m$. This value of χ depends on β, so that the condition of survivability poses a bound for β. Then the shakedown condition issuing from inequality (19) should be derived and the corresponding bound to β be established. Comparing this bound with the bound derived from the condition of integrity and a priori conditions, we obtain the interval for admissible values of β, for which both the shakedown and local integrity conditions hold.

In order to derive the conditions of overall integrity, the local bounds for β through the structure should be found and compared.

9. Example

Consider the structure shown in Fig. 1. The structure consists of three rods of the same cross-section area S and the same material. The lengths of the rods are in the relation: $l_1 = \beta l_2 = \beta l_3$, $\beta < 1$. The structure is loaded by a variable force $P(t)$ ranging in the interval $-P_1 \leqslant P \leqslant P_2$, $P_1 \leqslant P_2$, where $P(t)$ is a given function of time. The rods can experience only uniaxial tensile/compressive deformation.

The structure is to keep its integrity during a prescribed work time period t_w. Correspondingly, three sorts of problems can be considered. Firstly, if all the geometrical and loading parameters are given, then there is the problem to determine safe work time period t_w. Secondly, if t_w is prescribed, the problem is to determine safe bounds to loads p_1 and p_2 in such a way that the structure keeps its integrity during the period t_w. The third problem is the problem of geometrical design: to determine safe length of rod 1 (i.e., parameter β), for which the structure keeps its integrity during the period t_w for given values of $l_2 = l_3$ and P_1, P_2.

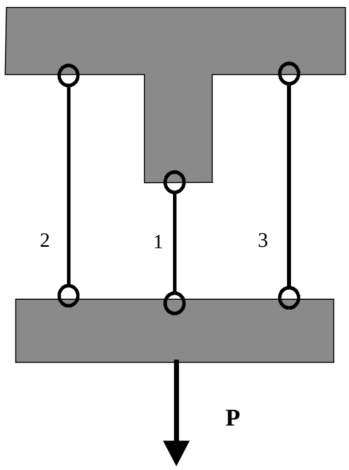

Fig. 1. The structure.

Due to symmetry, the strains and stresses in rods 2 and 3 are identical: $\varepsilon_2 = \varepsilon_3, \sigma_2 = \sigma_3$, and $\beta\varepsilon_1 = \varepsilon_2$. The stresses in the rods satisfy the equilibrium equation: $\sigma_1 + 2\sigma_2 = p$, where $-p_1 \leqslant p(t) \leqslant p_2$, $p = P(t)/S$, $p_1 = P_1/S$, $p_2 = P_2/S$.

It is assumed that the damage process is coupled with the process of viscoplastic deformation, i.e., the damage can develop only, if the viscoplastic deformation process is in progress. It is also assumed that the damage process starts simultaneously with the process of viscoplastic deformation, i.e., the damage threshold is small enough.

In the elastic undamaged state $\sigma_1 = p/(1+2\beta)$, $\sigma_2 = p\beta/(1+2\beta)$.

Suppose that rods 2 and 3 remain elastic, whereas rod 1 experiences viscoplastic deformation accompanied by damage. The yield condition of the rod material is taken in the form: $\bar{\Phi} = |\bar{\boldsymbol{\sigma}}| - \kappa(\chi) = 0$. Otherwise

$$\Phi = |\sigma| - (1-\Delta)\kappa(\chi) = 0, \tag{22}$$

where σ is the nominal stress, χ is the hardening parameter, and $\kappa(\chi)$ is the yield stress of undamaged material.

During the damage process, the current value of the unloading Young's modulus of rod 1 is $E_1 = E_0(1-\Delta)$ where E_0 is its initial value. At the same time, according to assumption, the Hooke moduli of rods 2 and 3 keep their initial values $E_2 = E_3 = E_0$. So that, after the damage process in rod 1 has started, the purely elastic response of the structure to the current value of the load $p(t)$ is $\sigma_1^{\mathrm{E}} = \sigma_2^{\mathrm{E}}(1-\Delta)/\beta = cp$, where $c = (1-\Delta)/(2\beta + (1-\Delta))$ and Δ is a current value of the damage parameter. Thus, the nominal stress in rod 1 can be represented as $\sigma_1 = cp + \rho$ where ρ is the residual stress in rod 1.

Now, the yield function of rod 1 can be rephrased as $\Phi = |cp + \rho| - (1-\Delta)\kappa(\chi)$ where ρ, χ and Δ are actual.

At the post-adaptation stage χ and Δ do not vary. Under fixed values of Δ and χ, the function $\xi = \Phi$ reaches its absolute maximum value under $p = p_2$, if $cp + \rho \geqslant 0$ and $\max \Phi = \Phi_2 = cp_2 + \rho - (1-\Delta)\kappa(\chi)$. However, if $cp + \rho \leqslant 0$, the yield function reaches its absolute maximum value under $p = p_1$ and $\max \Phi = \Phi_1 = cp_1 - \rho - (1-\Delta)\kappa(\chi)$. The function $\max \Phi$ is minimal, if $\Phi_1 = \Phi_2$. This equation provides $\rho = -c(p_2 - p_1)/2$. The corresponding value of the yield function is $\xi_{\mathrm{m}} \equiv \min \max \Phi = (p_1 + p_2)c/2 - (1-\Delta)\kappa(\chi)$.

Assume for simplicity that the deformation process starts from the undamaged state. According to Section 4, to obtain the shakedown condition, it is necessary to set $\Delta = \chi = 0$ in Φ, and require $\zeta_{\mathrm{m}} \leqslant 0$. This operation provides the inequality $(p_1 + p_2)/2(2\beta + 1) - \kappa(0) \leqslant 0$ that leads to the following shakedown condition

$$\frac{p_1 + p_2}{2} \leqslant \kappa(0)(2\beta + 1) = B_1. \tag{23}$$

Eq. (19) issues in the connection between limit values of Δ and $\chi : \Delta = 1 + 2\beta - \frac{p_1 + p_2}{2\kappa(\chi)}$. The maximal possible value of the hardening parameter (χ_{m}) is derived from (19) by setting $\Delta = \Delta_{\mathrm{c}}$.

$$\chi_{\mathrm{m}} = \phi\left(\frac{p_1 + p_2}{2}\,\frac{1}{2\beta + 1 - \Delta_{\mathrm{c}}}\right), \tag{24}$$

where $\chi = \phi(y)$ denotes the function inverse to the function $y = \kappa(\chi)$.

Condition of survivability (15) provides an upper bound to the safe value of designed work time period:

$$t_{\mathrm{w}} < \Delta_{\mathrm{c}}/vg(\chi_{\mathrm{m}}, \Delta_{\mathrm{c}}), \tag{25}$$

where $g(\chi, \Delta)$ is a given material function. $\boldsymbol{\sigma} : \ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} < \frac{m}{m-1}(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ See (11).

This inequality resolves all three possible problems. First of all it determines an upper bound to the structure lifetime duration. Further, the bounds to p_1, p_2 and β can be determined.

Take for simplicity that $g(\chi, \varDelta) \equiv \chi$. Then

$$t_{\mathrm{w}} < \frac{\varDelta_{\mathrm{c}}}{v\phi\left(\frac{p_1+p_2}{2}\frac{1}{2\beta+1-\varDelta_{\mathrm{c}}}\right)}. \tag{26}$$

Otherwise

$$\frac{p_1+p_2}{2} < (1-\varDelta_{\mathrm{c}}+2\beta)\kappa(\varDelta_{\mathrm{c}}/vt_{\mathrm{w}}) = B_2. \tag{27}$$

Inequalities (23) and (27) should be valid simultaneously. Their comparison provides us with a wanted condition of shakedown and integrity during the prescribed work time t_{w}. Eventually, the condition depends on the given parameters: if $\beta_1 < \beta_2$, then condition (23) holds; in the opposite case (27) is valid.

Inequality (25) provides also a possibility to resolve the problem of geometrical design. Again take for simplicity $g(\chi, \varDelta) \equiv \chi$. In this case

$$\beta > \frac{p_1+p_2}{4\kappa(\varDelta_{\mathrm{c}}/vt_{\mathrm{w}})} - \frac{1-\varDelta_{\mathrm{c}}}{2}. \tag{28}$$

Condition of shakedown (23) provides

$$\beta \geqslant \frac{p_1+p_2}{4\kappa(0)} - \frac{1}{2}. \tag{29}$$

Similar to the previous case the lower bound to β depends on the given parameters. As inequalities (28) and (29) should hold simultaneously, the inequality resulting in larger values of β has to be considered. But it should be remembered that according to the given condition it has to be $\beta < 1$.

Appendix A. Boundedness of dissipated energy (Koiter's inequality)

The boundedness of dissipated energy can be taken as a necessary and sufficient condition of shakedown (Debordes and Nayroles, 1976).

If $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ is not equal to zero identically, then the constitutive inequality (8) is strict. Following Koiter, it can be transformed into the form

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} < \frac{m}{m-1}(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \tag{A1}$$

where $m > 1$ is a number.

Integrating this inequality gives

$$W(T) = \int_{\omega}\int_0^T \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^{\mathrm{p}}\,\mathrm{d}t\,\mathrm{d}\omega < \frac{m}{m-1}\int_{\omega}\int_0^T (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \boldsymbol{\varepsilon}^{\mathrm{p}}\,\mathrm{d}t\,\mathrm{d}\omega, \tag{A2}$$

where ω is the volume of the structure, and T is an arbitrary time instant.

Consider the integral

$$A = \frac{1}{2}\int_Q (\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}) : \mathbf{L} : (\boldsymbol{\rho} - \hat{\boldsymbol{\rho}})\,\mathrm{d}\mathbf{Q} \geqslant 0, \tag{A3}$$

where $\boldsymbol{\rho}(\mathbf{x}, \mathrm{t})$ is the actual residual stress and $\hat{\boldsymbol{\rho}}(\mathbf{x})$ is a time-independent residual stress such that the stress $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{\mathrm{E}} + \hat{\boldsymbol{\rho}}$ is safe: $\Phi(\hat{\boldsymbol{\sigma}}, \chi) \leqslant 0$.

Notice that if the virtual stress $\hat{\boldsymbol{\sigma}}(\mathbf{x}, \mathrm{t})$ is safe with respect to the initial yield surface, then it is safe with respect to subsequent yield surfaces, until the initial yield surface is located in the interior of current yield surfaces, otherwise coincides with one of them. This condition is fulfilled not only at the stage of material hardening where current yield surfaces encompass the preceded ones, but also at a part of the material softening stage adjacent to the hardening saturation point. In other words, inequality (8) holds not only for ascending branch of the strain–stress curve where the material hardens, but also for a part of the descending branch where the material softens.

The derivative of A with respect to time is equal to

$$\dot{A} = \int_{\omega} \dot{\boldsymbol{\rho}} : \ \mathbf{L} : (\boldsymbol{\rho} - \hat{\boldsymbol{\rho}})\,\mathrm{d}\omega + \frac{1}{2}\int_{\omega} (\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}) : \ \dot{\mathbf{L}} : (\boldsymbol{\rho} - \hat{\boldsymbol{\rho}})\,\mathrm{d}\omega.$$

As $\boldsymbol{\varepsilon}^{\mathrm{re}} = \mathbf{L}{:}\boldsymbol{\rho}$, then $\dot{\boldsymbol{\rho}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{re}} - \dot{\mathbf{L}} : \ \boldsymbol{\rho}$. After obvious transformations, $\dot{A}$ can be reduced to the form

$$\dot{A} = -\int_{\omega} (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}\omega - \frac{1}{2}\int_{\omega} \boldsymbol{\rho} : \ \dot{\mathbf{L}} : \ \boldsymbol{\rho}\,\mathrm{d}\omega + \frac{1}{2}\int_{\omega} \hat{\boldsymbol{\rho}} : \ \dot{\mathbf{L}} : \ \hat{\boldsymbol{\rho}}\,\mathrm{d}\omega.$$

Integrating this equality over time provides

$$A(T) - A(0) = -\int_0^T\int_{\omega} (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}t\,\mathrm{d}\omega - \frac{1}{2}\int_0^T\int_{\omega} \boldsymbol{\rho} : \ \dot{\mathbf{L}} : \ \boldsymbol{\rho}\,\mathrm{d}t\,\mathrm{d}\omega + \frac{1}{2}\int_0^T\int_{Q} \hat{\boldsymbol{\rho}} : \ \dot{\mathbf{L}} : \ \hat{\boldsymbol{\rho}}^{r}\,\mathrm{d}t\,\mathrm{d}Q. \tag{A4}$$

Comparing (A2) with (A4), the following inequality is obtained:

$$W(T) + \frac{1}{2}\frac{m}{m-1}\int_0^T\int_{\omega} \boldsymbol{\rho} : \ \dot{\mathbf{L}} : \ \boldsymbol{\rho}\,\mathrm{d}w\,\mathrm{d}t \leqslant \frac{m}{m-1}(A(0) - A(T)) + \frac{1}{2}\frac{m}{m-1}\int_0^T\int_{\omega} \hat{\boldsymbol{\rho}} : \dot{\mathbf{L}} : \ \hat{\boldsymbol{\rho}}\,\mathrm{d}t\,\mathrm{d}\omega.$$

Taking into account inequalities (6) and $A(T) \geqslant 0$, the above inequality can be reduced to the form

$$W(T) \leqslant \frac{m}{m-1}A(0) + \frac{1}{2}\frac{m}{m-1}\int_0^T\int_{\omega} \hat{\boldsymbol{\rho}} : \dot{\mathbf{L}} : \ \hat{\boldsymbol{\rho}}\,\mathrm{d}t\,\mathrm{d}\omega. \tag{A5}$$

It follows from (A5) that the total dissipation (W) is bounded independently on the magnitude of T. This is the required result.

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